Journal of Mathematical Analysis and Applications **233,** 56–76 (1999)
Article ID jmaa.1998.6236, available online at http://www.idealibrary.com on IDEAL®

Existence and Decay to Contact Problems for Thermoviscoelastic Plates*

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Submitted by Colin Rogers

Received February 10, 1998

We consider the dynamic contact problem for thermoviscoelastic plates. We show the existence of weak solutions as well as exponential decay of the solution.

Key Words: thermoviscoelasticity; exponential decay.

1. INTRODUCTION

Problems involving thermoelastic contact arise naturally in many situations, particularly those involving industrial processes when two or more materials may come in contact or may lose contact as a result of thermoelastic expansion or contraction. Such thermoelastic phenomena can be divided into three parts: static, quasistatic, and full dynamic.

The quasistatic and static cases with various boundary conditions have been widely studied [1, 2, 5, 7, 12, 17–19], both numerically and theoretically. Various kinds of existence, uniqueness, and stability results are established. These papers contain a variety of linear and nonlinear boundary condition, but in each case the problem involves both a single tempera-

* Supported by a grant of CNPq (Brazil).

0022-247X/99 $30.00

ture and a single displacement, so that reformulation leads to one nonlinear equation for a single temperature.

In contrast, the fully dynamic problem is different from that of the quasistatic case. The quasistatic system can be viewed as being of mixed elliptic–parabolic type, while the dynamic case is of mixed hyperbolic–parabolic type. The latter is more complicated, and we have only a few results concerning existence and uniqueness.

Mathematically this problem is an example of a variational inequality, where a constraint is imposed on the unknown function u rather than on the time derivative of u.

In spite of the obvious importance of the subject in the applications, there are relatively few mathematical results about general problems of plate contact. A general variational inequality model was derived in Duvaut and Lions [8], where evolution problems with unilateral conditions were considered. The existence of weak solution was proved under the assumption that the body and the rigid foundation are in constant contact.

The wave equation with Signorini's contact condition was considered by Kim [3]. Using the penalty method, together with the compensated compactness method, Kim derived the variational inequality, which is equivalent to the contact problem.

Recently, Shi and Shillor [13] proved the existence of a solution to the n-dimensional quasistatic thermoelastic contact problem, provided the coefficient of thermal expansion is sufficiently small. The one-dimensional quasistatic problem of thermoelastic contact was considered in a series of papers by Gilbert et al. [12], Shi and Shillor [17, 18], Shi et al. [19], and by Copetti and Elliot [5]. The problem was formulated as a fully coupled variational inequality in [12] and the existence of strong solution was established. Numerical aspects of the problem were considered in [19] and in [5]. In particular, Copetti and Elliot obtained error estimates using the finite element method and proved the existence of periodic solutions. The foregoing papers provide no information on the asymptotic behavior of the solution.

Let us denote by Ω an open bounded set of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We assume that the boundary is divided into two parts:

$$\partial\Omega = \Gamma_0 \cup \Gamma_1, \qquad \overline{\Gamma}_0 \cap \Gamma_1 = \emptyset, \qquad \Gamma_0 \text{ with positive measure}.$$

Let us denote by $\nu = (\nu_1, \nu_2)$ the unitary external normal of $\partial\Omega$ and by $\tau = (-\nu_2, \nu_1)$ the unitary tangent positively oriented. Let μ be a positive constant, $0 < \mu < 1$. Let α be a positive constant and M be a real function defined on $[0, \infty[$. The aim of this paper is to study the contact problem for thermoviscoelastic plates. The system in question is

$$u_{tt} + \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u - \Delta u_t + \alpha\,\Delta\theta = 0 \quad \text{in } \Omega\times]0,T[, \tag{1.1}$$

$$\theta_t - \Delta\theta - \alpha\,\Delta u_t = 0 \quad \text{in } \Omega\times]0,T[, \tag{1.2}$$

with the boundary conditions

$$\theta = 0 \quad \text{on } \partial\Omega\times]0,T[, \tag{1.3}$$

$$u = 0, \qquad \frac{\partial u}{\partial\nu} = 0 \quad \text{on } \Gamma_0\times]0,T[, \tag{1.4}$$

$$u \geq 0, \qquad \frac{\partial u}{\partial\nu} \geq 0 \quad \text{on } \Gamma_1\times]0,T[, \tag{1.5}$$

$$\mathscr{B}_1 u \geq 0, \qquad -\mathscr{B}_2 u + M\left(\int_\Omega |\nabla u|^2\,dx\right)\frac{\partial u}{\partial\nu} + \frac{\partial u_t}{\partial\nu} - \alpha\frac{\partial\theta}{\partial\nu} \geq 0 \quad \text{on } \Gamma_1\times]0,T[, \tag{1.6}$$

$$(\mathscr{B}_1 u)\frac{\partial u}{\partial\nu} = 0 \quad \text{on } \Gamma_1\times]0,T[, \tag{1.7}$$

$$\left\{-\mathscr{B}_2 u + M\left(\int_\Omega |\nabla u|^2\,dx\right)\frac{\partial u}{\partial\nu} + \frac{\partial u_t}{\partial\nu} - \alpha\frac{\partial\theta}{\partial\nu}\right\}u = 0 \quad \text{on } \Gamma_1\times]0,T[\tag{1.8}$$

and initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \theta(x,0) = \theta_0(x) \quad \text{in } \Omega, \tag{1.9}$$

where

$$\mathscr{B}_1 u := \Delta u + (1-\mu)B_1 u,$$

$$\mathscr{B}_2 u := \frac{\partial\,\Delta u}{\partial\nu} + (1-\mu)\frac{\partial B_2\, u}{\partial\tau},$$

with the operators B_1 and B_2 given by

$$B_1 u := 2\nu_1\nu_2\frac{\partial^2 u}{\partial x\,\partial y} - \nu_1^2\frac{\partial^2 u}{\partial y^2} - \nu_2^2\frac{\partial^2 u}{\partial x^2},$$

$$B_2 u := \left(\nu_1^2 - \nu_2^2\right)\frac{\partial^2 u}{\partial x\,\partial y} + \nu_1\nu_2\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right).$$

Here we assume that M is a C^1 function satisfying

$$-\infty < \int_0^\infty M(\tau)\, d\tau. \tag{1.10}$$

Note that the function $\hat{M}$,

$$\hat{M}(s) := \int_0^s M(\tau)\, d\tau + \hat{M}_0, \quad \text{with } -\hat{M}_0 := \inf_{\eta \in [0,\infty[} \int_0^\eta M(\tau)\, d\tau,$$

is nonnegative.

The preceding system models the vertical deflection u of the middle plane of the plate from its equilibrium position, which, due to the difference of temperature θ, may come in contact or may lose contact with a rigid foundation as a result of a thermoelastic expansion or contraction. For plates we have a complicated boundary conditions because the contact can be produced as a result of bending moment about the tangent or the normal vector. By α we denote a thermal parameter and by μ the Poisson ratio.

We will show that there exists a weak solution that decays to zero exponentially as time goes to infinity. We now give a brief outline of this work. In the next section we show the existence of a weak solution to the obstacle problem. To do this we use the penalized method. That is, we define an equation depending on a parameter ε, whose solutions are regular enough to apply multiplicative techniques. Then we show that when $\epsilon \to 0$, the corresponding solution converges to the solution of the obstacle problem. To secure estimates for the penalized problem we have to construct a special basis of $H^4(\Omega)$. Finally, in Section 3 we show the exponential decay of the solution for both the penalized and the contact problem.

2. EXISTENCE OF SOLUTIONS

Let us introduce the subspace V of $H^2(\Omega)$ given by

$$V := \left\{ w \in H^2(\Omega) : w = 0,\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\}.$$

Let us define the bilinear form

$$a(u,v) := \int_\Omega \left\{ \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + \mu\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} \right) + 2(1-\mu)\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x\,\partial y} \right\} dx.$$

By Korn's inequality, we have that the functional $\sqrt{a(\cdot,\cdot)}$ defines a norm in V equivalent to the norm of $H^2(\Omega)$. In this conditions we can establish the following lemma due to Duvaut and Lions [4].

LEMMA 2.1. *Let us suppose that u and v are functions of $H^4(\Omega) \cap V$. Then we have*

$$\int_\Omega (\Delta^2 u) v\, dx = a(u,v) + \int_{\Gamma_1} \left\{ (\mathscr{B}_2 u) v - (\mathscr{B}_1 u) \frac{\partial v}{\partial \nu} \right\} d\Gamma_1.$$

See [4] for the proof.

Next we summarize a result owing to Kim [3].

LEMMA 2.2. *Let us denote by $\{u_k\}$ a sequence of function such that*

$$u^k \overset{*}{\rightharpoonup} u \quad \text{weak star in } L^\infty\big(0,T; H^\beta(\Omega)\big),$$
$$u_t^k \rightharpoonup u_t \quad \text{weak in } L^2(0,T; H^\alpha(\Omega))$$

when $k \to \infty$, for $\alpha < \beta$. Then we have

$$u^k \to u \quad \text{strong in } C([0,T]; H^r(\Omega))$$

for any $r < \beta$.

See [3] for the proof.

In that follows we will define what we understand for weak solution. Let us denote by K the convex subset of V and by W the subspace of $H^1(\Omega)$ given by

$$K := \left\{ w \in V : w \geq 0, \frac{\partial w}{\partial \nu} \geq 0 \text{ on } \Gamma_1 \right\},$$
$$W := \left\{ w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_0 \right\}.$$

DEFINITION 2.1. The couple (u, θ) is a weak solution to the system (1.1)–(1.9) when

$$u \in L^\infty(0,T;K),$$
$$u_t \in L^\infty\big(0,T; L^2(\Omega)\big) \cap L^2(0,T;W) \cap C\big([0,T]; H^{-1/2}(\Omega)\big),$$
$$\theta \in L^\infty\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H_0^1(\Omega)\big),$$
$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \theta(x,0) = \theta_0(x) \quad \text{in } \Omega$$

and also

$$\begin{aligned}
&\langle u_t(T), v(T) - u(T)\rangle_{H^{-1/2}(\Omega)\times H^{1/2}(\Omega)} - \int_\Omega u_1(v(0) - u_0)\,dx \\
&\quad - \int_0^T \int_\Omega u_t(v_t - u_t)\,dx\,dt + \int_0^T a(u, v - u)\,dt \\
&\quad + \int_0^T M\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega \nabla u(\nabla v - \nabla u)\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \nabla u_t(\nabla v - \nabla u)\,dx\,dt - \alpha\int_0^T \int_\Omega \nabla\theta(\nabla v - \nabla u)\,dx\,dt \geq 0,
\end{aligned} \tag{2.1}$$

$$-\int_0^T \int_\Omega \theta z_t\,dx\,dt + \int_0^T \int_\Omega \nabla\theta\,\nabla z\,dx\,dt + \alpha\int_0^T \int_\Omega \nabla u_t \nabla z\,dx\,dt = 0 \tag{2.2}$$

for any $v \in L^\infty(0, T; K)$, such that $v_t \in L^\infty(0, T; L^2(\Omega))$, and $z \in C_0^\infty(\Omega \times]0, T[)$.

Remark 2.1. The regularity $u_t \in C([0, T]; H^{-1/2}(\Omega))$ is a consequence of the others regularities of u, u_t, and θ established in Definition 2.1 and inequality (2.1). In fact, taking $v = \pm\varphi + u$, where $\varphi \in C_0^\infty$, we easily see that u satisfies Eq. (1.1) in the sense of $\mathscr{D}'(\Omega \times]0, T[)$. Using the regularity of u and θ it follows that

$$u_{tt} \in L^2([0, T]; H^{-2}(\Omega)).$$

Applying Lemma 2.2 for u_{tt} and u_t instead of u_t and u we conclude that

$$u_t \in C([0, T]; H^r(\Omega)) \quad \text{for } r < 0,$$

in particular for $r = -\frac{1}{2}$.

Remark 2.2. Assuming more regularity for the couple (u, θ), Definition 2.1 is equivalent to system (1.1)–(1.9).

We first show the existence of solutions to the penalized problem. Then we show that such solutions converge to the solution of the system

(2.1)–(2.2). Let us fixed $\epsilon > 0$. The penalized system is given by

$$u^\epsilon_{tt} + \Delta^2 u^\epsilon - M\left(\int_\Omega |\nabla u^\epsilon|^2\,dx\right)\Delta u^\epsilon - \Delta u^\epsilon_t + \alpha\,\Delta\theta^\epsilon = 0 \quad \text{in } \Omega\times]0,T[, \tag{2.3}$$

$$\theta^\epsilon_t - \Delta\theta^\epsilon - \alpha\,\Delta u^\epsilon_t = 0 \quad \text{in } \Omega\times]0,T[, \tag{2.4}$$

with the boundary conditions

$$\theta^\epsilon = 0 \quad \text{on } \partial\Omega\times]0,T[, \tag{2.5}$$

$$u^\epsilon = 0, \qquad \frac{\partial u^\epsilon}{\partial\nu} = 0 \quad \text{on } \Gamma_0\times]0,T[, \tag{2.6}$$

$$\mathscr{B}_1 u^\epsilon = \frac{1}{\epsilon}\left[\frac{\partial u^\epsilon}{\partial\nu}\right]^- - \epsilon\frac{\partial u^\epsilon_t}{\partial\nu} \quad \text{on } \Gamma_1\times]0,T[, \tag{2.7}$$

$$-\mathscr{B}_2 u^\epsilon + M\left(\int_\Omega |\nabla u^\epsilon|^2\,dx\right)\frac{\partial u^\epsilon}{\partial\nu} + \frac{\partial u^\epsilon_t}{\partial\nu} - \alpha\frac{\partial\theta^\epsilon}{\partial\nu} = \frac{1}{\epsilon}[u^\epsilon]^- - \epsilon u^\epsilon_t \quad \text{on } \Gamma_1\times]0,T[\tag{2.8}$$

and the initial condition

$$u^\epsilon(x,0) = u^\epsilon_0(x), \qquad u^\epsilon_t(x,0) = u^\epsilon_1(x), \qquad \theta^\epsilon(x,0) = \theta^\epsilon_0 \quad \text{in } \Omega. \tag{2.9}$$

Remark 2.3. The variational formulation of (2.3)–(2.8) is given by

$$\begin{aligned}\int_\Omega u^\epsilon_{tt} w\,dx + a(u^\epsilon, w) = {} & -M\left(\int_\Omega |\nabla u^\epsilon|^2\,dx\right)\int_\Omega \nabla u^\epsilon\,\nabla w\,dx \\ & - \int_\Omega \nabla u^\epsilon_t\,\nabla w\,dx - \alpha\int_\Omega \nabla\theta^\epsilon\,\nabla w\,dx \\ & + \frac{1}{\epsilon}\int_{\Gamma_1}\left[\frac{\partial u^\epsilon}{\partial\nu}\right]^- \frac{\partial w}{\partial\nu}\,d\Gamma_1 - \epsilon\int_{\Gamma_1}\frac{\partial u^\epsilon_t}{\partial\nu}\frac{\partial w}{\partial\nu}\,d\Gamma_1 \\ & + \frac{1}{\epsilon}\int_{\Gamma_1}[u^\epsilon]^-\,w\,d\Gamma_1 - \epsilon\int_{\Gamma_1} u^\epsilon_t w\,d\Gamma_1,\end{aligned} \tag{2.10}$$

$$\int_\Omega \theta^\epsilon_t z\,dx + \int_\Omega \nabla\theta^\epsilon\,\nabla z\,dx + \alpha\int_\Omega \nabla u^\epsilon_t\,\nabla z\,dx = 0 \tag{2.11}$$

$\forall w \in V$, $\forall z \in H^1_0(\Omega)$.

Let us introduce the functionals

$$E(t,v,\phi) = \frac{1}{2}\left\{\int_\Omega |v_t|^2\,dx + a(v,v) + \int_\Omega |\phi|^2\,dx\right\},$$

$$\mathscr{E}_\epsilon(t,v,\phi) = \frac{1}{2}\left\{\int_\Omega |v_t|^2\,dx + a(v,v) + \hat{M}\left(\int_\Omega |\nabla v|^2\,dx\right) + \int_\Omega |\phi|^2\,dx \right.$$

$$\left. + \frac{1}{\epsilon}\int_{\Gamma_1}\left|\left[\frac{\partial v}{\partial \nu}\right]^-\right|^2 d\Gamma_1 + \frac{1}{\epsilon}\int_{\Gamma_1}\left|[v]^-\right|^2 d\Gamma_1\right\},$$

where $\hat{M}$ is such that $\hat{M}' = M$ and is taken such that it is nonnegative over $[0,\infty[$.

Lemma 2.3. *Let M be a C^1 function over $[0,\infty[$ satisfying the condition* (1.10). *If the initial data satisfy*

$$u_0^\epsilon \in H^4(\Omega)\cap H_0^2(\Omega), \qquad u_1^\epsilon \in H_0^2(\Omega), \qquad \theta_0^\epsilon \in H^2(\Omega)\cap H_0^1(\Omega),$$

then there exists one unique solution to the system (2.9)–(2.11) *satisfying*

$$u^\epsilon \in L^\infty(0,T;V), \qquad u_t^\epsilon \in L^\infty(0,T;V),$$

$$\theta^\epsilon \in L^\infty\big(0,T;H^2(\Omega)\cap H_0^1(\Omega)\big).$$

Proof. We first choose a convenient basis to $H^4(\Omega)\cap V$. Since $H^4(\Omega)\cap H_0^2(\Omega)$ is a closed subspace of $H^4(\Omega)\cap V$ with the usual norm, we can make the decomposition

$$H^4(\Omega)\cap V = \big(H^4(\Omega)\cap H_0^2(\Omega)\big)\oplus F,$$

where F is the orthogonal complement of $H^4(\Omega)\cap H_0^2(\Omega)$ with relation to the inner product of $H^4(\Omega)$ in the space $H^4(\Omega)\cap V$. To simplify notation, in what follows we omit the superscript ϵ in our calculations.

Let $\{w_i\colon i\in\mathbb{N}\}$ be a basis for the space $H^4(\Omega)\cap V$ such that $w_{2i-1}\in H^4(\Omega)\cap H_0^2(\Omega)$ and $w_{2i}\in F$ for any $i\in\mathbb{N}$ and also that the set $\{z_i\colon i\in\mathbb{N}\}$ is a basis of $H^2(\Omega)\cap H_0^1(\Omega)$. Let us denote by (u^{2m},θ^{2m}) the Galerkin approximations

$$u^{2m}(\cdot,t) = \sum_{i=1}^{2m} a_{i,m}(t)w_i(\cdot),$$

$$\theta^{2m}(\cdot,t) = \sum_{i=1}^{2m} b_{i,m}(t)z_i(\cdot)$$

that are the solutions of the system of ordinary differential equations

$$\begin{aligned}\int_\Omega u_{tt}^{2m} w\,dx + a(u^{2m}, w) &= -M\left(\int_\Omega |\nabla u^{2m}|^2\,dx\right)\int_\Omega \nabla u^{2m}\,\nabla w\,dx \\ &\quad - \int_\Omega \nabla u_t^{2m}\,\nabla w\,dx - \alpha\int_\Omega \nabla\theta^{2m}\,\nabla w\,dx \\ &\quad + \frac{1}{\epsilon}\int_{\Gamma_1}\left[\frac{\partial u^{2m}}{\partial\nu}\right]^- \frac{\partial w}{\partial\nu}\,d\Gamma_1 - \epsilon\int_{\Gamma_1}\frac{\partial u_t^{2m}}{\partial\nu}\,\frac{\partial w}{\partial\nu}\,d\Gamma_1 \\ &\quad + \frac{1}{\epsilon}\int_{\Gamma_1}[u^{2m}]^-\,w\,d\Gamma_1 - \epsilon\int_{\Gamma_1} u_t^{2m} w\,d\Gamma_1,\end{aligned} \tag{2.12}$$

$$\int_\Omega \theta_t^{2m} z\,dx + \int_\Omega \nabla\theta^{2m}\,\nabla z\,dx + \alpha\int_\Omega \nabla u_t^{2m}\,\nabla z\,dx = 0 \tag{2.13}$$

$\forall w \in V_{2m} := \operatorname{span}\{w_1, \dots, w_{2m}\}$, $\forall z \in H_{2m} := \operatorname{span}\{z_1, \dots, z_{2m}\}$, and with the initial conditions

$$u^{2m}(\cdot, 0) = u^{0,2m}, \qquad u_t^{2m}(\cdot, 0) = u^{1,2m}, \qquad \theta^{2m}(\cdot, 0) = \theta^{0,2m},$$

where

$$\begin{aligned} u^{0,2m} &:= \text{projection of } u^0 \text{ on } V_{2m} \cap H_0^2(\Omega), \\ u^{1,2m} &:= \text{projection of } u^1 \text{ on } V_{2m} \cap H_0^2(\Omega), \\ \theta^{0,2m} &:= \text{projection of } \theta^0 \text{ on } H_{2m}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} u^{0,2m} &\to u^0 \quad \text{in } H^4(\Omega) \cap H_0^2(\Omega), \\ u^{1,2m} &\to u^1 \quad \text{in } H_0^2(\Omega), \\ \theta^{0,2m} &\to \theta^0 \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

The existence of solution is guaranteed by the Picard theorem. Our next step is to establish the a priori estimates. Taking $w = u_t^{2m}$ in (2.12) and $z = \theta^{2m}$ in (2.13) we get

$$\begin{aligned}\frac{d}{dt}\mathscr{E}_\epsilon(t, u^{2m}, \theta^{2m}) &= -\int_\Omega |\nabla u_t^{2m}|^2\,dx - \int_\Omega |\nabla\theta^{2m}|^2\,dx - \epsilon\int_{\Gamma_1}\left|\frac{\partial u_t^{2m}}{\partial\nu}\right|^2 d\Gamma_1 \\ &\quad - \epsilon\int_{\Gamma_1} |u_t^{2m}|^2\,d\Gamma_1.\end{aligned}$$

Integrating with respect to the time and keeping in mind that the initial data are bounded for $\epsilon > 0$ fixed, the following estimates hold:

$$u^{2m} \text{ is bounded in } L^\infty(0,T;V), \tag{2.14}$$

$$u_t^{2m} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W), \tag{2.15}$$

$$\theta^{2m} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2\big(0,T;H_0^1(\Omega)\big), \tag{2.16}$$

$$[u^m]^-, \left[\frac{\partial u^{2m}}{\partial \nu}\right]^- \text{ are bounded in } L^\infty\big(0,;L^2(\Gamma_1)\big), \tag{2.17}$$

$$u_t^{2m}, \frac{\partial u_t^{2m}}{\partial \nu} \text{ are bounded in } L^2\big(0,T;L^2(\Gamma_1)\big). \tag{2.18}$$

To get the second order estimate, let us differentiate the equations (2.12) and (2.13) with relation to t. Taking $w = u_{tt}^{2m}$ and $z = \theta_t^{2m}$, respectively, we arrive at

$$\frac{d}{dt}E\big(t,u_t^{2m},\theta_t^{2m}\big) = -\int_\Omega |\nabla u_{tt}^{2m}|^2\,dx - \int_\Omega |\nabla \theta_t^{2m}|^2\,dx - \epsilon\int_{\Gamma_1}\left|\frac{\partial u_{tt}^{2m}}{\partial \nu}\right|^2 d\Gamma_1$$

$$- \epsilon\int_{\Gamma_1} |u_{tt}^{2m}|^2\,d\Gamma_1 + R_1 + R_2 + R_3, \tag{2.19}$$

where

$$R_1 := -2M'\left(\int_\Omega |\nabla u^{2m}|^2\,dx\right)\int_\Omega \nabla u^{2m}\,\nabla u_t^{2m}\,dx\int_\Omega \nabla u^{2m}\,\nabla u_{tt}^{2m}\,dx,$$

$$R_2 := -M\left(\int_\Omega |\nabla u^{2m}|^2\,dx\right)\int_\Omega \nabla u_t^{2m}\,\nabla u_{tt}^{2m}\,dx,$$

$$R_3 := +\frac{1}{\epsilon}\int_{\Gamma_1}\left[\frac{\partial u^{2m}}{\partial \nu}\right]_t^- \frac{\partial u_{tt}^{2m}}{\partial \nu}\,d\Gamma_1 + \frac{1}{\epsilon}\int_{\Gamma_1}[u^{2m}]_t^-\,u_{tt}^{2m}\,d\Gamma_1.$$

Using Young's inequality together with relationships (2.14), (2.15), and (2.18) we get

$$\int_0^T \{R_1 + R_2 + R_3\}\,d\tau \le c_0(\epsilon) + \frac{1}{2}\int_0^T\int_\Omega |\nabla u_{tt}^{2m}|^2\,dx\,d\tau$$

$$+ \frac{\epsilon}{2}\int_0^T\int_{\Gamma_1}\left|\frac{\partial u_{tt}^{2m}}{\partial \nu}\right|^2 d\Gamma_1\,d\tau$$

$$+ \frac{\epsilon}{2}\int_0^t\int_{\Gamma_1}|u_{tt}^{2m}|^2\,d\Gamma_1\,d\tau, \tag{2.20}$$

where c_0 does not depend on m. Taking $t = 0, w = u_{tt}^{2m}(0)$, and $z = \theta_t^{2m}(0)$ in (2.12) and (2.13) and integrating by parts we get that $u_{tt}^{2m}(0)$ and $\theta_t^{2m}(0)$ are bounded. Integrating (2.19) with respect to the time variable and keeping in mind (2.20), we get

$$E\big(t, u_t^{2m}, \theta_t^{2m}\big) \leq -\frac{1}{2}\int^T \int_\Omega |\nabla u_{tt}^{2m}|^2\,dx\,d\tau - \int_0^T \int_\Omega |\nabla \theta_t^{2m}|^2\,dx\,d\tau$$
$$- \frac{\epsilon}{2}\int_0^T \int_{\Gamma_1} \left|\frac{\partial u_{tt}^{2m}}{\partial \nu}\right|^2 d\Gamma_1\,d\tau - \frac{\epsilon}{2}\int_0^T \int_{\Gamma_1} |u_{tt}^{2m}|^2\,d\Gamma_1\,d\tau$$
$$+ c_0(\epsilon) + E\big(0, u_t^{2m}, \theta_t^{2m}\big),$$

whence it follows that

$$u_t^{2m} \text{ is bounded in } L^\infty(0, T; V),$$
$$u_{tt}^{2m} \text{ is bounded in } L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2(0, T; W),$$
$$\theta_t^{2m} \text{ is bounded in } L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2\big(0, T; H_0^1(\Omega)\big),$$
$$u_{tt}^{2m}, \frac{\partial u_{tt}^{2m}}{\partial \nu} \text{ are bounded in } L^2\big(0, T; L^2(\Gamma_1)\big).$$

From the preceding estimates there exists a subsequence, which we still denote the same way, and function u and θ such that

$$u^{2m} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, T; V),$$
$$u_t^{2m} \overset{*}{\rightharpoonup} u_t \quad \text{in } L^\infty(0, T; V),$$
$$\theta^{2m} \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty\big(0, ; H^2(\Omega) \cap H_0^1(\Omega)\big).$$

From Lemma 2.2 we have

$$u^{2m} \to u \quad \text{in } C\big([0, T]; H^{2-\delta}(\Omega)\big) \tag{2.21}$$

for any $\delta > 0$. Taking $\delta = 1$ we conclude that

$$M\left(\int_\Omega |\nabla u^{2m}|^2\,dx\right)u^{2m} \to M\left(\int_\Omega |\nabla u|^2\,dx\right)u \quad \text{in } C\big([0, T]; H^1(\Omega)\big).$$

Taking $\delta = 1/2$ and using the trace theorem, we conclude that

$$\left[\frac{\partial u^{2m}}{\partial \nu}\right]^- \to \left[\frac{\partial u}{\partial \nu}\right]^- \quad \text{in } C\big([0,T]; L^2(\partial\Omega)\big),$$

$$[u^{2m}]^- \to [u]^- \quad \text{in } C\big([0,T]; H^1(\partial\Omega)\big).$$

The rest of the proof of the existence is a matter of routine. To prove the uniqueness let us suppose that there exists two solutions (u^1, θ^1) and (u^2, θ^2) of system (2.9)–(2.11), denoted by

$$U = u^1 - u^2, \qquad \Theta = \theta^1 - \theta^2.$$

We have that U and Θ satisfy

$$\begin{aligned}
&\int_\Omega U_{tt} w\,dx + a(U,w)\\
&\quad = -M\left(\int_\Omega |\nabla u^1|^2\,dx\right)\int_\Omega \nabla U\,\nabla w\,dx\\
&\qquad -\left\{M\left(\int_\Omega |\nabla u^1|^2\,dx\right) - M\left(\int_\Omega |\nabla u^2|^2\,dx\right)\right\}\int_\Omega \nabla u^2\,\nabla w\,dx\\
&\qquad -\int_\Omega \nabla U_t\,\nabla w\,dx - \alpha\int_\Omega \nabla\Theta\,\nabla w\,dx\\
&\qquad +\frac{1}{\epsilon}\int_{\Gamma_1}\left[\frac{\partial u^1}{\partial \nu}\right]^- \frac{\partial w}{\partial \nu}\,d\Gamma_1 - \frac{1}{\epsilon}\int_{\Gamma_1}\left[\frac{\partial u^2}{\partial \nu}\right]^- \frac{\partial w}{\partial \nu}\,d\Gamma_1\\
&\qquad +\frac{1}{\epsilon}\int_{\Gamma_1}[u^1]^-\,w\,d\Gamma_1 - \frac{1}{\epsilon}\int_{\Gamma_1}[u^2]^-\,w\,d\Gamma_1\\
&\qquad -\epsilon\int_{\Gamma_1}\frac{\partial U_t}{\partial \nu}\frac{\partial w}{\partial \nu}\,d\Gamma_1 - \epsilon\int_{\Gamma_1} U_t w\,d\Gamma_1,
\end{aligned} \tag{2.22}$$

$$\int_\Omega \Theta_t z\,dx + \int_\Omega \nabla\Theta\,\nabla z\,dx + \alpha\int_\Omega \nabla U_t\,\nabla z\,dx = 0, \tag{2.23}$$

$$U(x,0) = 0, \qquad U_t(x,0) = 0, \qquad \Theta(x,0) = 0 \tag{2.24}$$

for any $w \in V$ and $z \in H_0^1(\Omega)$. Taking $w = U_t$ in (2.22) and $z = \Theta$ in (2.23), we get

$$\begin{aligned}
\frac{d}{dt}E(t,U,\Theta) = &-\int_\Omega |\nabla U_t|^2\,dx - \epsilon\int_{\Gamma_1}\left|\frac{\partial U_t}{\partial \nu}\right|^2 d\Gamma_1 - \epsilon\int_{\Gamma_1}|U_t|^2\,d\Gamma_1\\
&+ I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{2.25}$$

where

$$I_1 := -M\left(\int_\Omega |\nabla u^1|^2\,dx\right)\int_\Omega \nabla U\,\nabla U_t\,dx,$$

$$I_2 := \left\{M\left(\int_\Omega |\nabla u^1|^2\,dx\right) - M\left(\int_\Omega |\nabla u^2|^2\,dx\right)\right\}\int_\Omega \nabla u^2\,\nabla U_t\,dx,$$

$$I_3 := \frac{1}{\epsilon}\int_{\Gamma_1}\left\{\left[\frac{\partial u^1}{\partial \nu}\right]^- - \left[\frac{\partial u^2}{\partial \nu}\right]^-\right\}\frac{\partial U_t}{\partial \nu}\,d\Gamma_1,$$

$$I_4 := \frac{1}{\epsilon}\int_{\Gamma_1}\left\{[u^1]^- - [u^2]^-\right\}U_t\,d\Gamma_1.$$

Since $t \to M(\int_\Omega |\nabla u^1|^2\,dx)$ is a C^1 function, using Young's inequality we conclude that

$$I_1 \le C_1\int_\Omega |\nabla U|^2\,dx + \tfrac{1}{4}\int_\Omega |\nabla U_t|^2\,dx,$$

$$\begin{aligned} I_2 &\le C_2\left|\left(\int_\Omega |\nabla u^1|^2\,dx\right)^{1/2} - \left(\int_\Omega |\nabla u^2|^2\,dx\right)^{1/2}\right| \\ &\quad \times \left(\int_\Omega |\nabla u^2|^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla U_t|^2\,dx\right)^{1/2} \\ &\le C_3\left(\int_\Omega |\nabla U|^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla U_t|^2\,dx\right)^{1/2} \\ &\le C_4\left(\int_\Omega |\nabla U|^2\,dx\right) + \tfrac{1}{4}\left(\int_\Omega |\nabla U_t|^2\,dx\right) \end{aligned}$$

for some positive constants $C_1, \dots, C_4$. Since the function $\lambda \to \lambda^+$ is Lipschitz we get

$$\begin{aligned} I_3 &\le \frac{1}{\epsilon}\int_{\Gamma_1}\left|\frac{\partial U}{\partial \nu}\right|\left|\frac{\partial U_t}{\partial \nu}\right| d\Gamma_1 \\ &\le \frac{1}{2\epsilon^3}\int_{\Gamma_1}\left|\frac{\partial U}{\partial \nu}\right|^2 d\Gamma_1 + \frac{\epsilon}{2}\int_{\Gamma_1}\left|\frac{\partial U_t}{\partial \nu}\right|^2 d\Gamma_1, \\ I_4 &\le \frac{1}{\epsilon}\int_{\Gamma_1} |U|\,|U_t|\,d\Gamma_1 \\ &\le \frac{1}{2\epsilon^3}\int_{\Gamma_1} |U|^2\,d\Gamma_1 + \frac{\epsilon}{2}\int_{\Gamma_1} |U_t|^2\,d\Gamma_1. \end{aligned}$$

Using the estimates on I_1, I_2, I_3, and I_4 in (2.25) we get

$$\frac{d}{dt}E(t,U,\Theta) \le C_\epsilon E(t,U,\Theta),$$

where C_ϵ is a positive constant. From Gronwall's inequality and (2.24), we get that

$$U \equiv 0, \qquad \Theta \equiv 0,$$

whence our conclusion follows. ∎

Now we are in a position to prove the existence of weak solutions to the variational system (2.1)–(2.2).

THEOREM 2.1. *Let M: $[0,\infty[\to \mathbb{R}$ be a C^1 function satisfying condition* (1.10). *Then for any initial data satisfying*

$$u^0 \in H_0^2(\Omega), \qquad u^1 \in L^2(\Omega), \qquad \theta^0 \in L^2(\Omega),$$

there exists at least one weak solution of system (1.1)–(1.9).

Proof. Let us consider the sequence (u_0^ϵ) in $H^4(\Omega) \cap H_0^2(\Omega)$, (u_1^ϵ) in $H_0^2(\Omega)$, and (θ_0^ϵ) in $H^2(\Omega) \cap H_0^1(\Omega)$, such that

$$u_0^\epsilon \to u_0 \quad \text{in } H_0^2(\Omega),$$

$$u_1^\epsilon \to u_1 \quad \text{in } L^2(\Omega),$$

$$\theta_0^\epsilon \to \theta_0 \quad \text{in } L^2(\Omega),$$

when $\epsilon \to 0$. Taking $w = u_t^\epsilon$ in (2.10) and $z = \theta^\epsilon$ in (2.11) we have

$$\frac{d}{dt}\mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon) = -\int_\Omega |\nabla u_t^\epsilon|^2\,dx - \int_\Omega |\nabla \theta^\epsilon|^2\,dx - \epsilon\int_{\Gamma_1}\left|\frac{\partial u_t^\epsilon}{\partial \nu}\right|^2 d\Gamma_1$$

$$- \epsilon\int_{\Gamma_1} |u_t^\epsilon|^2\,d\Gamma_1.$$

Integrating with respect to time and since $\mathscr{E}_\epsilon(0, u^\epsilon, \theta^\epsilon)$ is bounded, the following estimates hold:

$$u^\epsilon \text{ is bounded in } L^\infty(0,T;V), \tag{2.26}$$

$$u_t^\epsilon \text{ is bounded in } L^\infty\big(0,T;L^2(\Omega)\big) \in L^2(0,T;W), \tag{2.27}$$

$$\theta^\epsilon \text{ is bounded in } L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H_0^1(\Omega)\big), \tag{2.28}$$

$$\frac{1}{\sqrt{\epsilon}}[u^\epsilon]^-, \frac{1}{\sqrt{\epsilon}}\left[\frac{\partial u^\epsilon}{\partial \nu}\right]^- \text{ are bounded in } L^\infty\big(0,T;L^2(\Gamma_1)\big), \tag{2.29}$$

$$\sqrt{\epsilon}\,u_t^\epsilon, \sqrt{\epsilon}\,\frac{\partial u_t^\epsilon}{\partial \nu} \text{ are bounded in } L^2\big(0,T;L^2(\Gamma_1)\big). \tag{2.30}$$

So, we have that there exists a subsequence, which we still denote in the same way, such that

$$u^\epsilon \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;V), \tag{2.31}$$

$$u_t^\epsilon \overset{*}{\rightharpoonup} u_t \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big), \tag{2.32}$$

$$u^\tau \rightharpoonup u_t \quad \text{in } L^2(0,T;W), \tag{2.33}$$

$$\theta^\epsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big), \tag{2.34}$$

$$\theta^\epsilon \rightharpoonup \theta \quad \text{in } L^2\big(0,T;H_0^1(\Omega)\big). \tag{2.35}$$

Let us suppose that $v \in L^\infty(0,T,K)$ is such that $v_t \in L^\infty(0,T;L^2(\Omega))$. Let us consider $w = v - u^\epsilon$ in (2.10). Integrating by parts over $]0,T[$ we get

$$\begin{aligned}
&\int_\Omega u_t^\epsilon(T)(v(T) - u^\epsilon(T))\,dx - \int_\Omega u_1^\epsilon(v(0) - u_0^\epsilon)\,dx \\
&\quad - \int_0^T\int_\Omega u_t^\epsilon(v_t - u_t^\epsilon)\,dx\,dt + \int_0^T a(u^\epsilon, v - u^\epsilon)\,dt \\
&\quad + \int_0^T M\left(\int_\Omega |\nabla u^\epsilon|^2\,dx\right)\int_\Omega \nabla u^\epsilon(\nabla v - \nabla u^\epsilon)\,dx\,dt \\
&\quad + \int_0^T\int_\Omega \nabla u_t^\epsilon(\nabla v - \nabla u^\epsilon)\,dx\,dt - \alpha\int_0^T\int_\Omega \nabla\theta^\epsilon(\nabla v - \nabla u^\epsilon)\,dx\,dt \\
&\quad = \frac{1}{\epsilon}\int_0^T\int_{\Gamma_1}\left[\frac{\partial u^\epsilon}{\partial \nu}\right]^-\left(\frac{\partial v}{\partial \nu} - \frac{\partial u^\epsilon}{\partial \nu}\right)d\Gamma_1\,dt \\
&\qquad - \epsilon\int_0^T\int_{\Gamma_1}\frac{\partial u_t^\epsilon}{\partial \nu}\left(\frac{\partial v}{\partial \nu} - \frac{\partial u^\epsilon}{\partial \nu}\right)d\Gamma_1\,dt \\
&\qquad + \frac{1}{\epsilon}\int_0^T\int_{\Gamma_1}[u^\epsilon]^-(v - u^\epsilon)\,d\Gamma_1\,dt - \epsilon\int_0^T\int_{\Gamma_1}u_t^\epsilon(v - u^\epsilon)\,d\Gamma_1\,dt.
\end{aligned}$$

Since $v \in L^\infty(0,T;K)$, we have that $v(t) \geq 0$ and $(\partial v/\partial\nu)(t) \geq 0$ almost everywhere in $]0,T[$. It is easy to see that

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^T\int_{\Gamma_1}\left[\frac{\partial u^\epsilon}{\partial \nu}\right]^-\left(\frac{\partial v}{\partial \nu} - \frac{\partial u^\epsilon}{\partial \nu}\right)d\Gamma_1\,dt \\
&\qquad + \frac{1}{\epsilon}\int_0^T\int_{\Gamma_1}[u^\epsilon]^-(v - u^\epsilon)\,d\Gamma_1\,dt \geq 0,
\end{aligned}$$

whence we arrive at

$$\begin{aligned}
&\int_\Omega u_t^\epsilon(T)(v(T) - u^\epsilon(T))\,dx - \int_\Omega u_1^\epsilon(v(0) - u_0^\epsilon)\,dx \\
&\quad - \int_0^T \int_\Omega u_t^\epsilon(v_t - u_t^\epsilon)\,dx\,dt + \int_0^T a(u^\epsilon, v - u^\epsilon)\,dt \\
&\quad + \int_0^T M\left(\int_\Omega |\nabla u^\epsilon|^2\,dx\right)\int_\Omega \nabla u^\epsilon(\nabla v - \nabla u^\epsilon)\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \nabla u_t^\epsilon(\nabla v - \nabla u^\epsilon)\,dx\,dt - \alpha\int_0^T \int_\Omega \nabla\theta^\epsilon(\nabla v - \nabla u^\epsilon)\,dx\,dt \\
&\qquad \geq -\epsilon\int_0^T \int_{\Gamma_1} \frac{\partial u_t^\epsilon}{\partial \nu}\left(\frac{\partial v}{\partial \nu} - \frac{\partial u^\epsilon}{\partial \nu}\right)d\Gamma_1\,dt - \epsilon\int_0^T \int_{\Gamma_1} u_t^\epsilon(v - u^\epsilon)\,d\Gamma_1\,dt.
\end{aligned} \tag{2.36}$$

Using (2.30) we have

$$\begin{gathered}
\sqrt{\epsilon}\int_0^T \int_{\Gamma_1}\left(\sqrt{\epsilon}\,\frac{\partial u_t^\epsilon}{\partial \nu}\right)\left(\frac{\partial v}{\partial \nu} - \frac{\partial u^\epsilon}{\partial \nu}\right)d\Gamma_1\,dt \to 0, \\
\sqrt{\epsilon}\int_0^T \int_{\Gamma_1}\left(\sqrt{\epsilon}\,u_t^\epsilon\right)(v - u^\epsilon)\,d\Gamma_1\,dt \to 0
\end{gathered} \tag{2.37}$$

when $\epsilon \to 0$. From (2.3) we conclude that

$$u_{tt}^\epsilon \rightharpoonup u_{tt} \quad \text{in } L^2\big(0,T;H^{-2}(\Omega)\big).$$

From the convergences (2.31)–(2.32) and from Lemma 2.2, it follows that

$$\begin{aligned}
u^\epsilon &\to u \quad \text{in } C\big([0,T];H^{2-\delta}(\Omega)\big), && (2.38)\\
u_t^\epsilon &\to u_t \quad \text{in } C\big([0,T];H^{-1/2}(\Omega)\big)
\end{aligned}$$

for any $\delta > 0$. Taking $\delta = 3/2$, we get

$$\int_\Omega u_t^\epsilon(T)(w(T) - u^\epsilon(T))\,dx \to \big\langle u_t(T), w(T) - u(T)\big\rangle_{H^{-1/2}(\Omega)\times H^{1/2}(\Omega)}. \tag{2.39}$$

Taking $\delta = 1$, from the weak convergences we have

$$\int_0^T M\left(\int_\Omega |\nabla u^\epsilon|^2\,dx\right)\int_\Omega |\nabla u^\epsilon|^2\,dx\,dt \to \int_0^T M\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega |\nabla u|^2\,dx\,dt, \tag{2.40}$$

$$\int_0^T\int_\Omega \nabla u_t^\epsilon\,\nabla u^\epsilon\,Dx\,dt \to \int_0^T\int_\Omega \nabla u_t\,\nabla u\,dx\,dt, \tag{2.41}$$

$$\alpha\int_0^T\int_\Omega \nabla\theta^\epsilon\,\nabla u^\epsilon\,dx\,dt \to \alpha\int_0^T\int_\Omega \nabla\theta\,\nabla u\,dx\,dt. \tag{2.42}$$

From (2.33) and (2.38) we get

$$u_t^\epsilon \to u_t \quad \text{in } L^2\big(0,T;L^2(\Omega)\big);$$

that is,

$$\int_0^T\int_\Omega |u_t^\epsilon|^2\,dx\,dt \to \int_0^T\int_\Omega |u_t|^2\,dx\,dt. \tag{2.43}$$

From the lower semicontinuity of the norm we arrive at

$$\liminf_{\epsilon\to 0}\int_0^T a(u^\epsilon,u^\epsilon)\,dt \geq \int_0^T a(u,u)\,dt,$$

which is equivalent to

$$\limsup_{\epsilon\to 0}\left\{-\int_0^T a(u^\epsilon,u^\epsilon)\,dt\right\} \leq -\int_0^T a(u,u)\,dt. \tag{2.44}$$

Taking lim sup when $\epsilon \to 0$ in both sides of inequality (2.36), recalling the convergences (2.37)–(2.43), and inequality (2.44), we get (2.1). Taking $\delta = 1/2$ in (2.38) and applying the trace theorem we get

$$\left[\frac{\partial u^\epsilon}{\partial \nu}\right]^- \to \left[\frac{\partial u}{\partial \nu}\right]^- \quad \text{in } C\big([0,T];L^2(\Gamma_1)\big),$$

$$[u^\epsilon]^- \to [u]^- \quad \text{in } C\big([0,T];H^1(\Gamma_1)\big).$$

From (2.29)–(2.30) we get

$$[u]^-,\left[\frac{\partial u}{\partial \nu}\right]^- = 0, \quad \text{in } \Gamma_1\ \forall t \in [0,T];$$

therefore,

$$u,\ \frac{\partial u}{\partial \nu} \geq 0, \quad \text{in } \Gamma_1\ \forall t \in [0,T].$$

The proof is now complete. ∎

3. EXPONENTIAL DECAY

In this section we will show that the weak solutions of system (1.1)–(1.9) decay exponentially to zero as time goes to infinity. Let us consider M a continuous function satisfying

$$0 \leq \int_0^s M(\tau)\,d\tau \leq sM(s) \qquad \forall s \in [0, \infty[. \tag{3.1}$$

Note that any nonnegative and nondecreasing function M satisfies the foregoing condition; that is

$$\hat{M}(s) \leq sM(s) \qquad \forall s \in [0, \infty[.$$

THEOREM 3.1. *Let M be a C^1 function satisfying* (3.1). *If the initial data satisfy*

$$u^0 \in H_0^2(\Omega), \qquad u^1 \in L^2(\Omega), \qquad \theta^0 \in L^2(\Omega),$$

then the weak solution of system (1.1)–(1.9) *decays exponentially as time goes to infinity*; *that is*, *there exist positive constants c and γ such that*

$$E(t, u, \theta) \leq cE(0, u, \theta)e^{-\gamma t}.$$

Proof. Let us take $w = u_t^\epsilon$ in (2.10) and $z = \theta^\epsilon$ in (2.11) to get

$$\frac{d}{dt}\mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon) = -\int_\Omega |\nabla u_t^\epsilon|^2\,dx - \int_\Omega |\nabla\theta^\epsilon|^2\,dx - \epsilon\int_{\Gamma_1}\left|\frac{\partial u_t^\epsilon}{\partial\nu}\right|^2 d\Gamma_1$$

$$- \epsilon\int_{\Gamma_1}|u_t^\epsilon|^2\,d\Gamma_1. \tag{3.2}$$

Let us denote by $\mathscr{R}_\epsilon$ the functional

$$\mathscr{R}_\epsilon(t, v) := \int_\Omega v_t v\,dx + \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \frac{\epsilon}{2}\int_{\Gamma_1}\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma_1 + \frac{\epsilon}{2}\int_{\Gamma_1}|v|^2\,d\Gamma_1.$$

Taking $w = u^\epsilon$ in (2.10) we have

$$\frac{d}{dt}\mathscr{R}_\epsilon(t, u^\epsilon) = \int_\Omega |u_t^\epsilon|^2\,dx - a(u^\epsilon, u^\epsilon) - M\left(\int_\Omega |\nabla u^\epsilon|^2\,dx\right)\int_\Omega |\nabla u^\epsilon|^2\,dx$$

$$+ \alpha\int_\Omega \nabla u^\epsilon\,\nabla\theta^\epsilon\,dx - \frac{1}{\epsilon}\int_{\Gamma_1}\left|\left[\frac{\partial u^\epsilon}{\partial\nu}\right]^-\right|^2 d\Gamma_1$$

$$- \frac{1}{\epsilon}\int_{\Gamma_1}\left|[u^\epsilon]^-\right|^2 d\Gamma_1.$$

Since $u_t^\epsilon \in W$, from Poincaré's inequality there exists a positive constant $c_1 > 0$ such that

$$\int_\Omega |u_t^\epsilon|^2 \, dx \leq c_1 \int_\Omega |\nabla u_t^\epsilon|^2 \, dx.$$

Using Young's inequality and Korn's inequality, we conclude that there exists $c_2 > 0$ such that

$$\alpha \int_\Omega \nabla u^\epsilon \, \nabla \theta^\epsilon \, dx \leq \tfrac{1}{2} a(u^\epsilon, u^\epsilon) + c_2 \int_\Omega |\nabla \theta^\epsilon|^2 \, dx.$$

From relationship (3.1), we arrive at

$$\frac{d}{dt} \mathscr{R}_\epsilon(t, u^\epsilon) \leq c_1 \int_\Omega |\nabla u_t^\epsilon|^2 \, dx + \left(c_2 + \frac{1}{2} \right) \int_\Omega |\nabla \theta^\epsilon|^2 \, dx - \mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon). \tag{3.3}$$

Since the norms $\| \cdot \|_{H^2(\Omega)}$ and $\sqrt{a(\cdot, \cdot)}$ are equivalent in V from the trace theorem, there exists a positive constant $c_3 > 0$, such that for $\epsilon \leq 1$ we have

$$\left| \mathscr{R}_\epsilon(t, u^\epsilon) \right| \leq c_3 \mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon). \tag{3.4}$$

Multiplying Eq. (3.2) by k large enough, from relationships (3.3) and (3.4) we get

$$\frac{d}{dt} \left\{ k \mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon) + \mathscr{R}_\epsilon(t, u^\epsilon) \right\} \leq -\mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon) \tag{3.5}$$

and

$$\frac{\kappa}{2} \mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon) \leq k \mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon) + \mathscr{R}_\epsilon(t, u^\epsilon) \leq 2\kappa \mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon). \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\frac{d}{dt} \left\{ k \mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon) + \mathscr{R}_\epsilon(t, u^\epsilon) \right\} \leq -\frac{1}{2\kappa} \left\{ k \mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon) + \mathscr{R}_\epsilon(t, u^\epsilon) \right\}.$$

Integrating with respect to the time variable and using Gronwall's inequality, we have

$$k \mathscr{E}_\epsilon(t, u^\epsilon, \theta^e) + \mathscr{R}_\epsilon(t, u^\epsilon) \leq \left\{ k \mathscr{E}_\epsilon(0, u^\epsilon, \theta^\epsilon) + \mathscr{R}_\epsilon(0, u^\epsilon) \right\} e^{-(1/2\kappa)t}.$$

Using relationship (3.6) we get

$$\mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon) \leq 4\mathscr{E}_\epsilon(0, u^\epsilon(0), \theta^\epsilon(0)) e^{-(1/2\kappa)t}.$$

Note that $E(t, u^\epsilon, \theta^\epsilon) \leq \mathscr{E}_\epsilon(t, u^\epsilon, \theta^\epsilon)$ and $E(0, u^\epsilon, \theta^\epsilon) = \mathscr{E}_\epsilon(0, u^\epsilon, \theta^\epsilon)$; therefore,

$$E(t, u^\epsilon, \theta^\epsilon) \leq 4E(0, u^\epsilon(0), \theta^\epsilon(0)) e^{-(1/2\kappa)t}.$$

From the lower semicontinuity of the energy and from the strong convergence of the initial data, we finally get

$$E(t, u, \theta) \leq 4E(0, u^0, \theta^0) e^{-(1/2\kappa)t},$$

which completes the proof. ∎

ACKNOWLEDGMENT

The authors express their appreciation to the referee for his valuable suggestion, which improved this paper.

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